

# A CHARACTERIZATION OF SPIN STRUCTURE ON REAL BOTT TOWERS

RAISA DSOUZA AND V. UMA

ABSTRACT. The main aim of this article is to give a necessary and sufficient condition for a real Bott tower to admit a spin structure and further to characterize the real Bott towers admitting spin structure in terms of their associated acyclic digraph.

## 1. INTRODUCTION

Bott towers are iterated fibre bundles with fibre at each stage being  $\mathbb{CP}^1$ . In particular they are smooth projective complex toric varieties. They were constructed in [1] by M.Grossberg and Y.Karshon who show that a Bott-Samelson variety can be deformed to a Bott tower. Apart from [1], the topology and geometry of these objects have been studied by Civan and Ray in [2] and [3]. In particular, in [2], Civan looks at these manifolds as a special class of smooth toric varieties and studies their construction in terms of *Bott numbers*.

Indeed, from the viewpoint of toric topology, Bott-towers can be seen to also have the structure of a quasi-toric manifold [4] with the quotient polytope being the  $n$ -dimensional cube  $I^n$  where  $n$  is the complex dimension of the Bott tower (see [5] and [6]).

There has also been a parallel study on the topology of real Bott towers. These manifolds are constructed as iterated  $\mathbb{RP}^1 = S^1$ -bundles and can be viewed as a special example of a small cover defined by Davis and Januszkiewicz in [4] (see for example [7], [5]). Moreover, the data of the characteristic function for this small cover is encoded by an upper triangular nilpotent matrix in  $M_n(\mathbb{Z}_2)$ . We call this matrix the *Bott matrix* and its entries  $c_{i,j}$   $1 \leq i < j \leq n$ , Bott numbers.

In [5], Choi, Masuda and Oum characterize real Bott towers up to affine diffeomorphism using the associated Bott matrix.<sup>1</sup> Further, they naturally associate an acyclic directed graph to a real Bott tower. Indeed this is a directed graph whose adjacency matrix is the Bott matrix. Furthermore, in [5, Lemma 4.1] they give a criterion for orientability and symplectic structure on the Bott tower in terms of the combinatorics of the associated digraph.

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<sup>1</sup>In [5], a Bott matrix is more generally defined as the conjugate of a strictly upper triangular binary matrix by a permutation matrix.

In the first main result, Theorem 3.2, of the present paper, we give a necessary and sufficient condition for a real Bott tower to admit a spin structure in terms of certain algebraic identities on  $c_{i,j}$ 's. We prove this by deriving a closed formula for the second Stiefel-Whitney class of these manifolds. In Section 3.1, we consider a more general real Bott manifold  $M(B)$ , where  $B = (b_{i,j}) \in \mathcal{B}(n)$  [5, Section 1, page 2], and give a necessary and sufficient condition in Theorem 3.10 for  $M(B)$  to admit a spin structure in terms of algebraic identities on the  $b_{i,j}$ 's. We wish to mention here that the spin structure of real Bott manifolds has been studied earlier by Gąsior in [8], where a necessary and sufficient condition has been given for  $M(B)$  to be spin. The main result [8, Theorem 1.2] follows as a corollary to Theorem 3.10 of the present paper (see Corollary 3.11).

In the second main result, Theorem 4.5, we translate the combinatorial conditions of Theorem 3.2 in the language of acyclic digraphs. This enables us to further characterize the existence of spin structure on these manifolds by means of the combinatorics of the associated digraph. The results in Section 4 are motivated from [5, Section 4].

**1.1. Notations and Conventions.** In this section we recall the definition of a Bott tower and fix some notations (see [1]).

A *Bott tower* is a smooth complete complex toric variety which is constructed iteratively as follows:

Let  $Y_1 = \mathbb{CP}^1$ . Let  $L_2$  be a holomorphic complex line bundle on  $\mathbb{CP}^1$ . We then let  $Y_2 = \mathbb{P}(\mathbf{1} \oplus L_2)$  where  $\mathbf{1}$  is the trivial line bundle on  $\mathbb{CP}^1$ . Then  $Y_2$  is a  $\mathbb{CP}^1$  bundle over  $\mathbb{CP}^1$  which is a Hirzebruch surface. We can iterate this process for  $2 \leq j \leq n$ , where at each step,  $L_j$  is a complex line bundle over  $Y_{j-1}$ , and the variety  $Y_j = \mathbb{P}(\mathbf{1} \oplus L_j)$  is a  $\mathbb{CP}^1$  bundle over  $Y_{j-1}$ . The variety  $Y_n$  thus obtained after  $n$ -steps is called an  $n$ -step Bott tower.

**Definition 1.1.** In fact an  $n$ -step Bott tower is a smooth complete toric variety of dimension  $n$  whose fan  $\Delta$  can be described as follows:

We take a collection of integers  $\{a_{i,j}\}$ ,  $1 \leq i < j \leq n$ . Let  $e_1, \dots, e_n$  be the standard basis vectors of  $\mathbb{R}^n$ . Let  $v_j = e_j$  for  $1 \leq j \leq n$ ,

$$v_{n+j} = -e_j + \sum_{k=j+1}^n a_{j,k} e_k$$

for  $1 \leq j \leq n-1$  and  $v_{2n} = -e_n$ . We define the fan  $\Delta$  in  $\mathbb{R}^n$  consisting of cones generated by the set of vectors in any sub collection of  $\{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$  which does not contain both  $v_i$  and  $v_{n+i}$  for  $1 \leq i \leq n$ .

**Definition 1.2.** We can also view a Bott tower as a quasi-toric manifold (see [4]) over the  $n$ -cube  $I^n$  which is a simple convex polytope of dimension  $n$ . If we index the  $2n$  facets of  $I^n$  by  $F_1, F_2, \dots, F_n, F_{n+1}, \dots, F_{2n}$ , then the characteristic function is defined on the collection of facets,  $\mathcal{F}$  to  $\mathbb{Z}^n$  as follows:  $\lambda(F_j) = e_j$  for  $1 \leq j \leq n$ ,

$$\lambda(F_{n+j}) = -e_j + \sum_{k=j+1}^n a_{j,k} \cdot e_{j+k}$$

for  $1 \leq j \leq n-1$  and  $\lambda(F_{2n}) = -e_n$ .

1.1.1. *Real Bott tower.* We shall call the real part of the  $n$ -step complex Bott tower as the real  $n$ -step Bott tower.

In particular,  $(Y_2)_{\mathbb{R}}$  is an  $\mathbb{RP}^1$  bundle over  $(Y_1)_{\mathbb{R}} = \mathbb{RP}^1$ . Iteratively we construct  $(Y_j)_{\mathbb{R}}$  as an  $\mathbb{RP}^1$  bundle over  $(Y_{j-1})_{\mathbb{R}}$  for  $2 \leq j \leq n$ . The real  $n$ -step Bott tower  $(Y_n)_{\mathbb{R}}$  is indeed the real toric variety associated to the fan  $\Delta$  described above (see [9, Section 2.4] and [10]).

**Definition 1.3.** As in the complex case we can also view  $(Y_n)_{\mathbb{R}}$  as a *small cover* over the simple convex polytope  $I^n$ , where the characteristic map  $\lambda$  is defined on the collection of facets,  $\mathcal{F}$  to  $\mathbb{Z}_2^n$  as follows:  $\lambda(F_j) = e_j$  for  $1 \leq j \leq n$ ,

$$\lambda(F_{n+j}) = e_j + \sum_{k=j+1}^n c_{j,k} \cdot e_k$$

for  $1 \leq j \leq n-1$  and  $\lambda(F_{2n}) = e_n$  where  $c_{i,j} = a_{i,j} \bmod 2$  for  $1 \leq i < j \leq n$ . Thus  $(Y_n)_{\mathbb{R}}$  is homeomorphic to the identification space  $\mathbb{Z}_2^n \times I^n / \sim$  where  $(t, p) \sim (t', p')$  if and only if  $p = p'$  and  $t \cdot (t')^{-1} \in G_{F(p)}$ . Here  $F(p) = F_1 \cap \dots \cap F_l$  is the unique face of  $I$  which contains  $p$  in its relative interior and  $G_{F(p)}$  is the rank- $l$  subgroup of  $\mathbb{Z}_2^n$  determined by the span of  $\lambda(F_1), \dots, \lambda(F_l)$ .

The topological structure of an  $n$ -step real Bott tower is completely determined by the simple convex polytope  $I^n$  and the data encoded by the matrix

$$(1.1) \quad C = (c_{i,j}) \in M_n(\mathbb{Z}_2)$$

where  $c_{i,j} = 0$  for  $i \geq j$ . Note that the  $i$ th row of  $C + I$  is  $\lambda(F_{n+i}) \in \mathbb{Z}_2^n$  for  $1 \leq i \leq n$ . We call  $C$  the *Bott matrix*. Thus  $(Y_n)_{\mathbb{R}}$  is the real Bott tower associated to  $C$ .

The 2-step real Bott tower is the torus or the Klein bottle depending on whether  $c_{1,2} = 0$  or  $c_{1,2} = 1$ . The 3-step real Bott tower is an  $\mathbb{RP}^1$  bundle over the torus or the Klein bottle whose topological structure depends on  $c_{1,2}, c_{1,3}$  and  $c_{2,3}$ .

**Note.** In this article, since we are mainly interested in the study of the real Bott tower, for notational simplicity we shall henceforth denote  $(Y_n)_{\mathbb{R}}$  by  $Y_n$ . If we wish to specify the associated Bott matrix we shall denote  $Y_n$  by  $Y_n(C)$ .

## 2. COHOMOLOGY OF $Y_n$

In this section we briefly recall the description of the cohomology ring with  $\mathbb{Z}_2$ -coefficients of  $Y_n$ . A discription of  $H^*(Y_n; \mathbb{Z}_2)$  has been given earlier in [7, Section 2]. We also recall the formula for total Stiefel-Whitney class as given in [4, Section 6].

**Proposition 2.1.** *Let  $\mathcal{R} := \mathbb{Z}_2[x_1, x_2, \dots, x_{2n}]$  and let  $\mathcal{I}$  denote the ideal in  $\mathcal{R}$  generated by the following set of elements*

$$(2.1) \quad \left\{ x_j x_{n+j}, x_1 + x_{n+1}, x_j + x_{n+j} + \sum_{i=1}^{j-1} c_{i,j} x_{n+i} \mid 2 \leq j \leq n \right\}$$

*As a graded  $\mathbb{Z}_2$ -algebra,  $H^*(Y_n; \mathbb{Z}_2)$  is isomorphic to  $\mathcal{R}/\mathcal{I}$ .*

Let  $w_k(Y_n)$  denote the  $k^{\text{th}}$  Stiefel-Whitney class of  $Y_n$  for  $0 \leq k \leq n$  with the understanding that  $w_0(Y_n) = 1$ . Then  $w(Y_n) = 1 + w_1(Y_n) + \dots + w_n(Y_n)$  is the total Stiefel-Whitney class of  $Y_n$ .

**Proposition 2.2.** *Under the isomorphism of proposition (2.1) between  $H^*(Y_n; \mathbb{Z}_2)$  and  $\mathcal{R}/\mathcal{I}$  we have the identification*

$$(2.2) \quad w(Y_n) = \prod_{j=1}^{2n} (1 + x_j)$$

*where  $x_j$  for  $1 \leq j \leq 2n$  satisfy the relations (2.1).*

**Lemma 2.3.** *We have the following expression for the total Stiefel-Whitney class of  $Y_n$  :*

$$(2.3) \quad w(Y_n) = \prod_{j=2}^n \left( 1 + \sum_{i=1}^{j-1} c_{i,j} \cdot x_{n+i} \right).$$

*Proof.* Equation (2.2) can be rewritten as,

$$(2.4) \quad w(Y_n) = \prod_{j=1}^n (1 + x_j + x_{n+j} + x_j \cdot x_{n+j}).$$

Further, using the relations (2.1) we can rewrite equation (2.4) to get (2.3).  $\square$

The next proposition has been proved in [7, Lemma 2.2] for real Bott towers. We state and prove it here for completeness. Also see [11] for orientability criterion for any small cover.

**Proposition 2.4.** *The real Bott tower  $Y_n$  is orientable if and only if the sum of entries in each row the Bott matrix  $C = (c_{i,j})$  are zero in  $\mathbb{Z}_2$ , that is,*

$$(2.5) \quad \sum_{j=1}^n c_{i,j} \equiv 0 \pmod{2} \text{ for every } 1 \leq i \leq n.$$

*Proof.* By equating degree one terms on either side of equation (2.3) we get,

$$(2.6) \quad w_1(Y_n) = \sum_{i=1}^{n-1} \left( \sum_{j=i+1}^n c_{i,j} \right) \cdot x_{n+i} = \sum_{i=1}^{n-1} \left( \sum_{j=1}^n c_{i,j} \right) \cdot x_{n+i}$$

where the second equality follows from the fact that  $c_{i,j} = 0$  for  $i \geq j$ .

The proposition then follows from the fact that a compact connected differentiable manifold  $M$ , is orientable if and only if  $w_1(M) = 0$  and that as a  $\mathbb{Z}_2$ -vector space,  $H^1(Y_n; \mathbb{Z}_2)$  is isomorphic to the subspace of  $\mathcal{R}/\mathcal{I}$  freely generated by  $x_{n+i}$ ,  $1 \leq i \leq n$ .  $\square$

### 3. SPIN STRUCTURE ON REAL BOTT TOWERS

In this section we give a necessary and sufficient condition in terms of the Bott numbers for a Bott tower to admit a spin structure (Theorem 3.2).

**Definition 3.1.** The *spinor group*  $\text{Spin}(n)$  (for  $n \geq 3$ ) is the connected double cover of the special orthogonal group  $SO(n)$ . There exists a short exact sequence of Lie groups

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \xrightarrow{\lambda} SO(n) \rightarrow 1.$$

An oriented Riemannian manifold  $X$  is said to admit a spin structure if the oriented frame bundle  $F$  associated to its tangent bundle, which is a principal  $SO(n)$ -bundle, lifts to a principal  $\text{Spin}(n)$ -bundle. More precisely, if there is a principal  $\text{Spin}(n)$ -bundle  $P$  on  $X$  that is a double cover of  $F$ .

It is further known that an  $SO(n)$ -bundle admits a spin structure if and only if its second Stiefel-Whitney class is zero ([12, Theorem 1.7, pg 86]). Using this criterion, we give a necessary and sufficient condition, in terms of algebraic identities in the Bott numbers, for an  $n$  - step orientable Bott tower to admit a spin structure.

**Theorem 3.2.** *The Bott tower  $Y_n$  admits a spin structure if and only if the following two conditions are satisfied :*

(1) *The row sums of the Bott matrix  $C = (c_{i,j})$  are even. That is, for every  $1 \leq i \leq n$ ,*

$$(3.1) \quad \sum_{j=1}^n c_{i,j} \equiv 0 \pmod{2}$$

(2) For every  $1 \leq j < k \leq n$ ,

$$(3.2) \quad \underbrace{\sum_{r=1}^n c_{j,r} c_{k,r}}_{P_{jk}} + c_{j,k} \cdot \underbrace{\sum_{\substack{r,s=1 \\ r < s}}^n c_{k,r} c_{k,s}}_{Q_{jk}} \equiv 0 \pmod{2}$$

*Proof.* Condition (1), which is precisely (2.5), says that  $Y_n$  is orientable. We know that an orientable manifold admits a spin structure if and only if its second Stiefel-Whitney class vanishes ([12, Theorem 1.7, pg 86]). We will now prove that this is equivalent to condition (2).

Since Theorem 2.1 gives an isomorphism of graded  $\mathbb{Z}_2$ -algebras, the degree 2 term of  $w(Y_n)$ , namely  $w_2(Y_n)$  can be identified with the degree 2 term of expression (2.3) which is the class of the following term in  $\mathcal{R}/\mathcal{I}$ :

$$(3.3) \quad \sum_{1 \leq j < k \leq n-1} \left( \sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} c_{k,s} \right) x_{n+j} x_{n+k} + \sum_{k=1}^{n-2} \left( \sum_{\substack{r,s=k+1 \\ r < s}}^n c_{k,r} c_{k,s} \right) x_{n+k}^2.$$

Further, by the relations (2.1) we obtain,

$$(3.4) \quad x_{n+k}^2 = \sum_{j=1}^{k-1} c_{j,k} \cdot x_{n+j} x_{n+k}, \quad \forall \quad 2 \leq k \leq n.$$

Also, from the identity (3.1) we have,

$$(3.5) \quad c_{n-1,n} = 0.$$

Now, by substituting (3.4) and (3.5) in (3.3), it follows that  $w_2(Y_n)$  can be identified with the class of the following term in  $\mathcal{R}/\mathcal{I}$ :

$$(3.6) \quad \sum_{1 \leq j < k \leq n-2} \left( \underbrace{\sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} c_{k,s}}_I + c_{j,k} \cdot \underbrace{\sum_{\substack{r,s=k+1 \\ r < s}}^n c_{k,r} c_{k,s}}_{II} \right) x_{n+j} x_{n+k}.$$

The expression  $I$  in (3.6) can be rewritten as follows :

$$\begin{aligned}
 \sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} c_{k,s} &= \sum_{r=j+1}^n c_{j,r} \left( \sum_{s=k+1}^n c_{k,s} - c_{k,r} \right) \\
 (3.7) \qquad &= \sum_{r=j+1}^n c_{j,r} \cdot \sum_{s=k+1}^n c_{k,s} - \sum_{r=j+1}^n c_{j,r} c_{k,r} \\
 &= \sum_{r=1}^n c_{j,r} c_{k,r} =: P_{jk}
 \end{aligned}$$

The last equality in (3.7) follows by condition (1) and the fact that  $c_{i,j} = 0$  for  $i \geq j$ .

Again, using  $c_{i,j} = 0$  for  $i \geq j$ , the expression  $II$  in (3.6) can be rewritten as,

$$(3.8) \qquad c_{j,k} \cdot \sum_{\substack{r,s=1 \\ r < s}}^n c_{k,r} c_{k,s} =: Q_{jk}.$$

Further, it follows by the definition of  $C$  and by (3.1) that,

$$(3.9) \qquad P_{j \, n-1} = P_{jn} = Q_{j \, n-1} = Q_{jn} = 0.$$

Thus by (3.7), (3.8) and (3.9) it follows that  $w_2(Y_n)$  can be identified with the class of the following term in  $\mathcal{R}/\mathcal{I}$ :

$$(3.10) \qquad \sum_{1 \leq j < k \leq n} (P_{jk} + Q_{jk}) x_{n+j} x_{n+k}.$$

Further, as a graded  $\mathbb{Z}_2$ -vector space  $H^2(Y_n; \mathbb{Z}_2)$  is isomorphic to the subspace of  $\mathcal{R}/\mathcal{I}$  freely generated over  $\mathbb{Z}_2$  by the classes of  $x_{n+j} x_{n+k}$ ,  $1 \leq j < k \leq n$ . Hence the theorem.  $\square$

**Remark 3.3.** The only oriented 2-step real Bott tower is the torus, which is classically known to be spin. The 3-step oriented real Bott towers  $Y_3$ , correspond to the following two Bott matrices :

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here, we immediately see that  $w_2(Y_3) = 0$  and hence  $Y_3$  admits a spin structure. This is a special case of the well known more general result of Steenrod that an oriented threefold is parallelizable.

**Example 3.4.** The 4-step Bott towers admitting spin structure correspond to the following list of associated Bott matrices :

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Remark 3.5.** Note that the above list of Bott matrices exhausts all orientable 4-step real Bott towers. Thus it follows that every orientable 4-step real Bott tower is also spin. Moreover, it is known that a 4-manifold  $M$ , is parallelizable if and only if it admits a spin structure (i.e  $w_1(M) = w_2(M) = 0$ ) and has vanishing Euler characteristic and signature ( $\chi(M) = \sigma(M) = 0$ ) (see [13, Section 4] and [14, p. 699]). Moreover, by Hirzebruch signature formula,  $\sigma(M) = \frac{1}{3}p_1(M)[M]$ , where  $p_1(M)$  is the first Pontrjagin class and  $[M]$  the fundamental class of  $M$ . Now, a real Bott tower has vanishing Euler characteristic (since  $\chi(Y_n) = \chi(Y_{n-1}) \cdot \chi(S^1)$  and  $\chi(S^1) = 0$ ) and vanishing Pontrjagin classes by ([4, Corollary 6.8 (i)]). Thus it follows that a 4-step real Bott tower is orientable if and only if it is parallelizable. Further, it corresponds to one of the eight Bott matrices in the above list.

The following example shows that this is not the case in dimensions 5 and higher. Indeed there are  $n$ -step Bott towers which are orientable but not spin when  $n \geq 5$ .

**Example 3.6.** Let  $Y_n$  be the  $n$ -step Bott tower,  $n \geq 5$ , associated to the Bott numbers  $c_{1,2} = 1$ ,  $c_{1,n-2} = 1$ ;  $c_{n-2,n-1} = 1$ ,  $c_{n-2,n} = 1$  and  $c_{i,j} = 0$  otherwise. These numbers clearly satisfy (3.1) but not (3.2). Indeed in this case, when  $j = 1$  and  $k = n - 2$ , the left hand side of (3.2) is  $c_{1,n-2} c_{n-2,n-1} c_{n-2,n} \equiv 1 \pmod{2}$ .

**Definition 3.7.** We call the Bott matrix  $C$  *spin* if and only if the associated Bott tower  $Y_n = Y_n(C)$  is spin.

**Definition 3.8.** Let  $R_i$  denote the  $i$ th row vector  $(0, \dots, 0, 0 = c_{i,i}, c_{i,i+1}, c_{i,i+2}, \dots, c_{i,n})$  of  $C$ . For every  $1 \leq j < k \leq n$ , we define another  $n \times n$  Bott matrix  $C_{jk}$  with  $R_j$  as the  $j$ th row and  $R_k$  as the  $k$ th row and remaining rows with all entries 0.



**Corollary 3.9.** *The Bott matrix  $C$  is spin if and only if  $C_{jk}$  is spin for every  $1 \leq j < k \leq n$ .*

*Proof.* From Theorem 3.2, a necessary and sufficient condition for  $C$  to be spin is that the entries  $c_{i,j}$ ,  $1 \leq j \leq n$  on the row  $R_i$  for every  $1 \leq i \leq n$  satisfy (3.1) and further, the entries  $c_{j,r}$ ,  $1 \leq r \leq n$  of  $R_j$  and  $c_{k,s}$ ,  $1 \leq s \leq n$  of  $R_k$  for every  $1 \leq j < k \leq n$  satisfy (3.2).

Again by Theorem 3.2, it follows that the necessary and sufficient condition for the Bott matrix  $C_{jk}$  to be spin is that the entries  $c_{j,r}$ ,  $1 \leq r \leq n$ , of the  $j$ th row and the entries  $c_{k,s}$ ,  $1 \leq s \leq n$ , of the  $k$ th row of  $C_{jk}$ , satisfy (3.1) and (3.2). This can be readily seen because any row of  $C_{jk}$ , other than the  $j$ th or  $k$ th row, has all entries as 0. Thus the entries on the  $i$ th row of  $C_{jk}$  where  $i \neq j, k$ , trivially satisfy (3.2). Moreover, if either  $i \neq j, k$  or  $l \neq j, k$  and  $1 \leq i < l \leq n$ , the entries of  $C_{jk}$ , on the  $i$ th and the  $l$ th row trivially satisfy (3.2). Hence the corollary.  $\square$

**3.1. Spin condition for a more generally defined real Bott manifold.** We recall here that Choi, Masuda and Oum give a more general definition of a Bott matrix. They call a square matrix  $B$  to be a Bott matrix if there exists a permutation matrix  $P$  and a strictly upper triangular binary matrix  $C$  such that  $B = PCP^{-1}$ . They denote by  $\mathcal{B}(n)$ , the set of all such  $n \times n$ -matrices [5]. Further, it follows from [5, Section 3] that  $B$  and  $C$  are *Bott equivalent*.

Moreover, in [5, Section 2] they also give a construction of a real Bott manifold  $M(B)$  associated to  $B$ . In particular, when  $B$  is a strictly upper triangular binary matrix then  $M(B)$  is nothing but the associated real Bott tower.

In the following theorem we give a necessary and sufficient condition for the Bott manifold  $M(B)$  to admit a spin structure where  $B$  is any matrix in  $\mathcal{B}(n)$ .

**Theorem 3.10.** *The real Bott manifold  $M(B)$  associated to  $B = (b_{i,j}) \in \mathcal{B}(n)$  admits a spin structure if and only if the entries  $b_{i,j}$  satisfy the following identities :*

(1) For  $1 \leq i \leq n$ ,

$$(3.11) \quad \sum_{j=1}^n b_{i,j} \equiv 0 \pmod{2}.$$

(2) For  $1 \leq j < k \leq n$ ,

$$(3.12) \quad \sum_{r=1}^n b_{j,r} b_{k,r} + b_{j,k} \cdot \sum_{\substack{r,s=1 \\ r < s}}^n b_{k,r} b_{k,s} \equiv 0 \pmod{2}.$$

*Proof.* Let  $B \in \mathcal{B}(n)$ . Let  $P$  be an  $n \times n$  permutation matrix such that  $B = PCP^{-1}$  for an  $n \times n$  strictly upper triangular binary matrix  $C$ . Let  $C = (c_{i,j})$  and  $\sigma \in S_n$  be the permutation corresponding to  $P$ . Note that by [5, (3.1)],

$$(3.13) \quad b_{\sigma(i),\sigma(j)} = c_{i,j} \quad \text{for } 1 \leq i, j \leq n$$

Now, by [5, Theorem 1.6], since  $B$  and  $C$  are Bott equivalent,  $M(B)$  and  $M(C)$  are affinely diffeomorphic. In particular,  $M(C)$  is spin if and only if  $M(B)$  is spin. The proof of the theorem now follows by substituting (3.13) in (3.1) and (3.2).  $\square$

We derive the following corollary, analogous to Corollary 3.9. We wish to remark here that this result has been proved in [8, Theorem 1.2] using different techniques. We omit the proof which is similiary to that of Corollary 3.9.

**Corollary 3.11.** *The Bott matrix  $B$  is spin if and only if  $B_{jk}$  is spin for every  $1 \leq j < k \leq n$ , where  $B_{jk}$  is the  $n \times n$  matrix having same  $j$ th and  $k$ th row as  $B$  and all other rows zero.*

**Remark 3.12.** Recall that, by [5, Table 1], [7, Section 7], [15, Section 3], any  $B \in \mathcal{B}(4)$  with  $M(B)$  orientable, is Bott equivalent to one of the following three Bott matrices :

$$(3.14) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, by [15, Section 3] and [5, Table 1], any  $B \in \mathcal{B}(5)$  with  $M(B)$  orientable, is Bott equivalent to one of the following eight Bott matrices :

$$(3.15) \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can readily check that all the three matrices in (3.14) are spin and only the first four matrices in (3.15) are spin.

Following [5, Table 1], we let  $\mathcal{Spin}_n$  denote the number of  $n$ -dimensional real spin Bott manifolds up to diffeomorphism. Then, by [5, Example 3.1], Remark 3.3 and Remark 3.12 above, we have the following table :

$n$	1	2	3	4	5
$\mathcal{O}_n$	1	1	2	3	8
$\mathcal{Spin}_n$	1	1	2	3	4

#### 4. DIGRAPH CHARACTERIZATION OF SPIN STRUCTURE ON REAL BOTT TOWERS

We begin this section by recalling the definition of an acyclic digraph associated to a real Bott tower. The dictionary between a real Bott tower and its associated acyclic digraph, via the Bott matrix, established in the work of Choi, Masuda and Oum [5, Section 4], gives a new way of studying the topology of these manifolds by means of the combinatorics of the associated digraph. The main theorem (Theorem 4.5) in this section gives a combinatorial criterion on the acyclic digraph, which characterizes the existence of spin structure on the associated real Bott tower.

**Definition 4.1.** A directed graph (digraph) is a tuple  $(V, E)$  consisting of a set  $V$ , of elements, called vertices and a set  $E$ , of ordered pairs of distinct vertices, called edges.

**Definition 4.2.** Let  $D = (V, E)$  be a digraph with *vertices*  $V = \{u_1, \dots, u_n\}$  and edges  $(u_i, u_j)$  indexed by an ordered pair of distinct vertices. In particular, we assume that  $D$  has no loops and has at most one directed edge between any pair of vertices. The *adjacency matrix*  $A(D)$  associated to  $D$  is therefore an  $n \times n$  matrix in  $M_n(\mathbb{Z}_2)$  with diagonal entries zero. Conversely, to any such matrix  $A \in M_n(\mathbb{Z}_2)$  we associate a graph with  $n$  vertices and an edge from  $u_i$  to  $u_j$  if and only if  $a_{i,j} = 1$ ,  $1 \leq i, j \leq n$ . In particular, given a Bott matrix  $C = (c_{i,j})$  (see 1.1) we can associate a digraph,  $D_C$  to it, having  $n$  vertices. Moreover, since the matrix is strictly upper triangular, the digraph  $D_C$  admits an ordering of vertices  $u_1, \dots, u_n$  such that  $i < j$  whenever there is an edge from  $u_i$  to  $u_j$ . In particular,  $D_C$  is an *acyclic digraph* (see [5, Section 4]).

**4.1. Notations :** Let  $D_C$  be an acyclic digraph associated to a Bott matrix  $C$ . For each vertex  $u_i$  of  $D_C$  we denote by,

$$N_{D_C}^+(u_i) := \{u_j \mid c_{i,j} = 1\} \quad \text{and} \quad N_i := |N_{D_C}^+(u_i)| \text{ is the out degree of } u_i$$

$$N_{D_C}^-(u_i) := \{u_j \mid c_{j,i} = 1\} \quad \text{and} \quad I_i := |N_{D_C}^-(u_i)| \text{ is the in degree of } u_i$$

(see [5, Section 4])

For each pair of vertices  $u_i, u_j$  we further denote by  $M_{ij} := |N_{D_C}^+(u_i) \cap N_{D_C}^+(u_j)|$ . More precisely,  $M_{ij}$  is the number of vertices  $u_k$  which are the out neighbours of both  $u_i$  and  $u_j$ .

The following two lemmas respectively reinterpret the terms  $P_{jk}$  and  $Q_{jk}$  on the left hand side of the identity (3.2), in terms of the combinatorial data of the digraph.

**Lemma 4.3.**

$$(4.1) \quad P_{jk} = M_{jk}$$

*Proof.* Note that the product  $c_{k,r} c_{j,r} \neq 0$  if and only if  $c_{k,r} = 1 = c_{j,r}$ , that is if and only if there is an edge from  $u_j$  to  $u_r$  as well as  $u_k$  to  $u_r$ . Thus the sum  $\sum_{r=1}^n c_{j,r} c_{k,r} = P_{jk}$  counts the number of vertices  $\{u_r\}$  that have edges from  $u_j$  as well as  $u_k$  coming into it. This number is precisely  $M_{jk}$ . Hence the lemma.  $\square$

**Lemma 4.4.**

$$(4.2) \quad Q_{jk} = c_{j,k} \cdot \binom{N_k}{2}$$

*Proof.* Note that the number of unordered pairs of distinct edges coming out of  $u_k$  in  $D_C$  is precisely  $\binom{N_k}{2}$ . Also, the product  $c_{k,r} c_{k,s} \neq 0$  if and only if  $c_{k,r} = 1 = c_{k,s}$ . Thus the sum  $\sum_{\substack{r,s=1 \\ r < s}}^n c_{k,r} c_{k,s} = Q_{jk}$  counts the total number of unordered pairs of distinct edges coming out of  $u_k$ . Hence the lemma.  $\square$

The next theorem reformulates Theorem 3.2 in terms of the associated digraph  $D_C$ .

**Theorem 4.5.** *The  $n$ -step Bott tower  $Y_n(C)$  admits a spin structure if and only if for the corresponding digraph  $D_C$  the following two conditions are true :*

- (1)  $N_k$  is even for all  $1 \leq k \leq n$ .
- (2)  $M_{jk}$  and  $c_{j,k} \cdot \binom{N_k}{2}$  have the same parity for all  $1 \leq j < k \leq n$ .

*Proof.* Note that, the out degree  $N_i$  of  $u_i$  is  $\sum_{j=1}^n c_{i,j}$ . Thus the identity (3.1) in the statement of Theorem 3.2 is equivalent to condition (1) above. Furthermore, it follows by Lemma 4.3 and Lemma 4.4 that the identity (3.2) in the statement of Theorem 3.2 is equivalent to condition (2) above. Hence the theorem.  $\square$

**Remark 4.6.** Condition (2) in Theorem 4.5 can be made more explicit as follows :

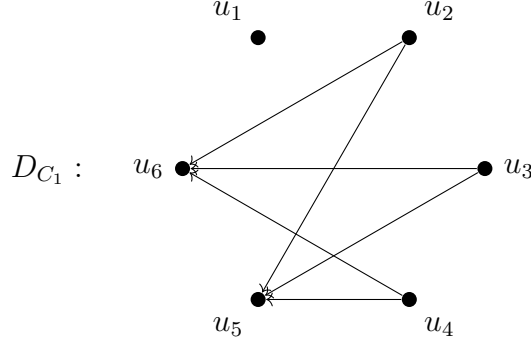
- (1) When  $N_k = 4m$  then  $\binom{N_k}{2}$  is always even. So condition (2) is equivalent to saying that  $M_{jk}$  is even.

- (2) When  $N_k = 4m - 2$  the  $\binom{N_k}{2}$  is always odd. So condition (2) is equivalent to saying that  $M_{jk}$  is even when there is no edge from  $u_j$  to  $u_k$  in  $D_C$  and  $M_{jk}$  is odd when there is an edge from  $u_j$  to  $u_k$  in  $D_C$ .

We will now look at some examples to illustrate Theorem 4.5.

**Example 4.7.**

$$(1) \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Here we have  $N_{D_{C_1}}^+(u_i) = \{u_5, u_6\}$  for  $i = 2, 3, 4$  and  $N_{D_{C_1}}^+(u_i) = \emptyset$  for  $i = 1, 5, 6$ .

Clearly,  $N_i = |N_{D_{C_1}}^+(u_i)|$  is even for all  $1 \leq i \leq 5$ .

Also,  $M_{jk} = |N_{D_{C_1}}^+(u_j) \cap N_{D_{C_1}}^+(u_k)| = 2$  for  $2 \leq j < k \leq 4$  and  $M_{jk} = 0$ , otherwise.

When  $j = 2$  and  $k = 3$  we get that  $M_{23} = 2$  and  $c_{2,3} \cdot \binom{N_3}{2} = 0$  have the same parity.

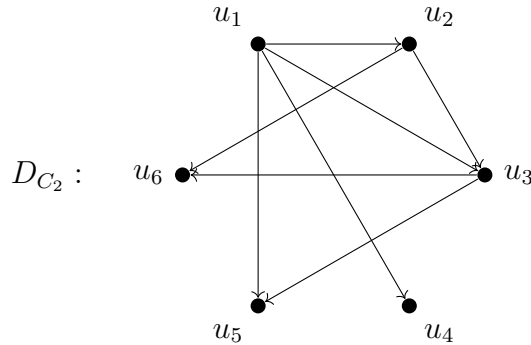
When  $j = 2$  and  $k = 4$  we get that  $M_{24} = 2$  and  $c_{2,4} \cdot \binom{N_4}{2} = 0$  have the same parity.

When  $j = 3$  and  $k = 4$  we get that  $M_{34} = 2$  and  $c_{3,4} \cdot \binom{N_4}{2} = 0$  have the same parity.

For other pairs  $j < k$  we get  $M_{jk} = 0$  and  $c_{j,k} \cdot \binom{N_k}{2} = 0$  have the same parity.

Thus the associated Bott tower admits a spin structure.

$$(2) \quad C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Here we have  $N_{D_{C_2}}^+(u_1) = \{u_2, u_3, u_4, u_5\}$ ,  $N_{D_{C_2}}^+(u_2) = \{u_3, u_6\}$ ,  $N_{D_{C_2}}^+(u_3) = \{u_5, u_6\}$  and  $N_{D_{C_2}}^+(u_i) = \emptyset$  for  $i = 4, 5, 6$ .

Clearly,  $N_i = |N_{D_{C_2}}^+(u_i)|$  is even for all  $1 \leq i \leq 6$ .

When  $j = 1$  and  $k = 2$  we have  $M_{12} = |N_{D_{C_2}}^+(u_1) \cap N_{D_{C_2}}^+(u_2)| = |\{u_3\}| = 1$  and  $c_{1,2} \cdot \binom{N_2}{2} = 1$  have the same parity.

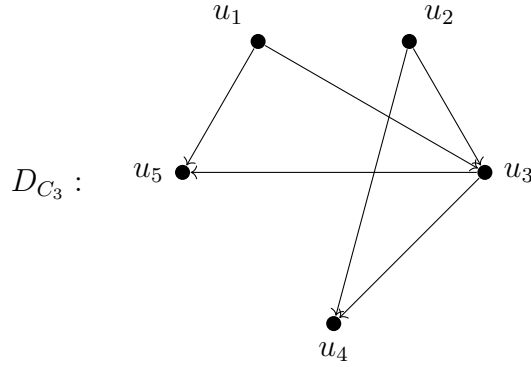
When  $j = 1$  and  $k = 3$  we have  $M_{13} = |N_{D_{C_2}}^+(u_1) \cap N_{D_{C_2}}^+(u_3)| = |\{u_5\}| = 1$  and  $c_{1,3} \cdot \binom{N_3}{2} = 1$  have the same parity.

When  $j = 2$  and  $k = 3$  we have  $M_{23} = |N_{D_{C_2}}^+(u_2) \cap N_{D_{C_2}}^+(u_3)| = |\{u_6\}| = 1$  and  $c_{2,3} \cdot \binom{N_3}{2} = 1$  have the same parity.

For other pairs  $j < k$  we have  $M_{jk} = 0$  and  $c_{j,k} \cdot \binom{N_k}{2} = 0$  have the same parity.

Thus the corresponding Bott tower admits a spin structure.

$$(3) \quad C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



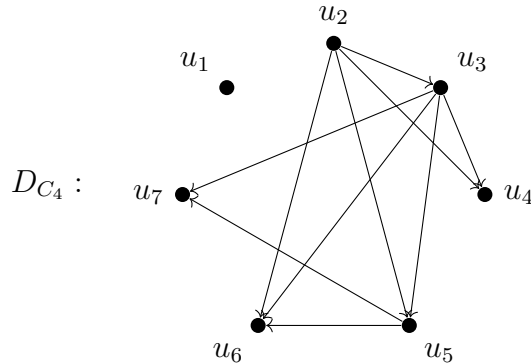
Here we have  $N_{D_{C_3}}^+(u_1) = \{u_3, u_5\}$ ,  $N_{D_{C_3}}^+(u_2) = \{u_3, u_4\}$ ,  $N_{D_{C_3}}^+(u_3) = \{u_4, u_5\}$  and  $N_{D_{C_3}}^+(u_i) = \emptyset$  for  $i = 4, 5$ .

Clearly,  $N_i = |N_{D_{C_3}}^+(u_i)|$  is even for all  $1 \leq i \leq 4$ .

When  $j = 1$  and  $k = 2$  we get  $M_{12} = |N_{D_{C_3}}^+(u_1) \cap N_{D_{C_3}}^+(u_2)| = |\{u_3\}| = 1$  and  $c_{1,2} \cdot \binom{N_2}{2} = 0$  do not have the same parity.

Thus the associated Bott tower does not admit a spin structure.

$$(4) \quad C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Here we have  $N_{D_{C_4}}^+(u_2) = \{u_3, u_4, u_5, u_6\}$ ,  $N_{D_{C_4}}^+(u_3) = \{u_4, u_5, u_6, u_7\}$ ,  $N_{D_{C_4}}^+(u_5) = \{u_6, u_7\}$  and  $N_{D_{C_4}}^+(u_i) = \emptyset$  for  $i = 1, 4, 6, 7$ .

Clearly,  $N_i = |N_{D_{C_4}}^+(u_i)|$  is even for all  $1 \leq i \leq 6$ .

When  $j = 2$  and  $k = 3$  we get  $M_{23} = |N_{D_{C_4}}^+(u_2) \cap N_{D_{C_4}}^+(u_3)| = |\{u_3, u_5, u_6\}| = 3$

and  $c_{2,3} \cdot \binom{N_3}{2} = 6$  do not have the same parity.

Thus the associated Bott tower does not admit a spin structure.

**4.2. Remark on higher Stiefel Whitney classes.** We hope to compute closed formulae for Stiefel-Whitney classes  $w_k(Y_n)$ , for  $k \geq 3$ , using Lemma 2.3 and also characterize their vanishing by looking at the corresponding digraph. In particular, we have the following result for  $k = n - 1$  :

**Theorem 4.8.** (1) *We have the following formula for  $w_{n-1}(Y_n)$  in  $H^*(Y_n; \mathbb{Z}_2)$  in terms of the Bott numbers  $c_{i,j}$ :*

$$w_{n-1}(Y_n) = c_{1,2} \cdot c_{2,3} \cdots c_{n-1,n} \cdot x_{n+1} \cdot x_{n+2} \cdots x_{2n-1}.$$

(2) *If  $Y_n$  is an oriented real Bott tower then  $w_{n-1}(Y_n) = 0$ .*

(3) *We have  $w_{n-1}(Y_n) = 0$  if and only if there exists a pair of vertices  $u_i, u_{i+1}$  in the digraph  $D_C$  with no edge from  $u_i$  to  $u_{i+1}$ .*

*Proof.* Using equation (2.3) we obtain the following recursive formula for  $w_k(Y_n)$  :

$$(4.3) \quad w_k(Y_n) = w_k(Y_{n-1}) + w_{k-1}(Y_{n-1}) \cdot \left( \sum_{i=1}^{n-1} c_{i,n} x_{n+i} \right)$$

for  $n \geq 2$  and  $1 \leq k \leq n$ . The proof of (1) now follows by induction on  $n$  and by using (4.3) and the fact that in  $H^*(Y_n; \mathbb{Z}_2)$  the following relations hold :

$$(4.4) \quad x_{n+1}^2 = 0; x_{n+1} \cdot x_{n+2}^2 = 0; x_{n+1} \cdot x_{n+2} \cdot x_{n+3}^2 = 0; \cdots; x_{n+1} \cdot x_{n+2} \cdots x_{2n-2}^2 = 0.$$

From (3.1),  $c_{n-1,n} = 0$  if  $Y_n$  is orientable. Hence (2) follows from (1). Also (3) follows from (1) and the definition of  $D_C$ .  $\square$

**Remark 4.9.** The assertion (2) of Theorem 4.8 is true for any even dimensional manifold but not in general true when the dimension is odd (see [16, Theorem II and examples on p. 94]). Our assertion although specific to the case of a real Bott tower, holds in all dimensions.

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DEPARTMENT OF MATHEMATICS, IIT MADRAS, CHENNAI, INDIA

*E-mail address*: raisadsouza1989@gmail.com; vuma@iitm.ac.in